# Cardano's Method

Cardano's method provides a technique for solving the general cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

in terms of radicals. As with the quadratic equation, it involves a "discriminant" whose sign determines the number (1, 2, or 3) of real solutions. However, its implementation requires substantially more technique than does the quadratic formula. For example, in the "irreducible case" of three real solutions, it calls for the evaluation of the cube roots of complex numbers.

In outline, Cardano's methods involves the following steps:

1. "Eliminate the square term" by the substitution $y = x + b/3a$. Rather than keeping track of such a substitution relative to the original cubic, the method often begins with an equation in the reduced form

$$x^3 + px + q = 0.$$

2. Letting $x = u+v$, rewrite the above equation as

$$u^3 + v^3 +(u+v)(3uv + p) + q = 0.$$

3. Setting $3uv + p = 0$, the above equation becomes $u^3 + v^3 = -q$. In this way, we obtain the system

$$u^3 + v^3 = -q$$

$$u^3v^3 = -p^3/27.$$

Since this system specifies both the sum and product of $u^3$ and $v^3$, it enables us to determine a quadratic equation whose roots are $u^3$ and $v^3$. This equation is

$$t^2 + qt -p^3/27$$

with solutions

$$u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}; \quad v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

In order to find u and v, we are now obligated to find the cube roots of these solutions. In the case

$$\mathbf{27q^2 + 4p^3 < 0}$$

this entails finding the cube roots of complex numbers.

Even in the case $27q^2 + 4p^3 > 0$, there are some unexpected wrinkles. These are illustrated by the equation

$$\mathbf{x^3 + x^2 - 2 = 0}$$

for which $x = 1$ is clearly a solution. Although Cardano's method enables one to find this root without confronting cube roots of complex numbers, it displays the solution $x = 1$ in the rather obscure form

$$\frac{\sqrt[3]{26+15\sqrt{3}} + \sqrt[3]{26-15\sqrt{3}}}{4}.$$

It is against this historical background that Chapter I of IADM develops an iteration-based alternative to Cardano's method at the pre-calculus level, one that is derived from "a Babylonian technique for finding cube roots."

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